British Mathematical Olympiad

Round 2 : Thursday, 11 February 1993

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than trying all four problems.*
- *The use of rulers and compasses is allowed, but calculators are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

Before March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (on 15-18 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team for this summer's International Mathematical Olympiad (to be held in Istanbul, Turkey, July 13–24) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session before leaving for Istanbul.

Do not turn over until **told to do so.**

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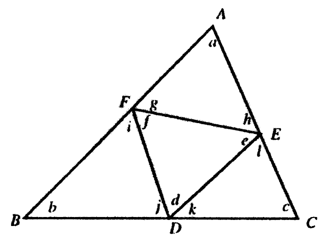
1. We usually measure angles in degrees, but we can use any other unit we choose. For example, if we use $30°$ as a new unit, then the angles of a $30°$, $60°$, $90°$ triangle would be equal to 1, 2, 3 new units respectively.

   The diagram shows a triangle ABC with a second triangle DEF inscribed in it. All the angles in the diagram are whole number multiples of some new (unknown unit); their sizes $a, b, c, d, e, f, g, h, i, j, k, \ell$ with respect to this new angle unit are all distinct.

   Find the smallest possible value of $a+b+c$ for which such an angle unit can be chosen, and mark the corresponding values of the angles a to ℓ in the diagram.

2. Let $m = (4^p - 1)/3$, where p is a prime number exceeding 3. Prove that 2^{m-1} has remainder 1 when divided by m.

3. Let P be an internal point of triangle ABC and let α, β, γ be defined by

   $$\alpha = \angle BPC - \angle BAC, \;\; \beta = \angle CPA - \angle CBA, \;\; \gamma = \angle APB - \angle ACB.$$

   Prove that

   $$PA\,\frac{\sin \angle BAC}{\sin \alpha} = PB\,\frac{\sin \angle CBA}{\sin \beta} = PC\,\frac{\sin \angle ACB}{\sin \gamma}.$$

4. The set $Z(m,n)$ consists of all integers N with mn digits which have precisely n ones, n twos, n threes, $\ldots$, n ms. For each integer $N \in Z(m,n)$, define $d(N)$ to be the sum of the absolute values of the differences of all pairs of consecutive digits. For example, $122313 \in Z(3,2)$ with $d(122313) = 1+0+1+2+2 = 6$. Find the average value of $d(N)$ as N ranges over all possible elements of $Z(m,n)$.